

# AVERAGE NUMBER OF SQUARES DIVIDING $mn$

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**ABSTRACT.** We study the asymptotic behaviour of  $\sum_{m,n \leq x} \tau_{1,2}(mn)$ , where  $\tau_{1,2}(n) = \sum_{ab^2=n} 1$ , using multidimensional Perron formula and complex integration method. An asymptotic formula with an error term  $O(x^{10/7})$  is obtained.

## 1. INTRODUCTION

Let  $f$  be a multiplicative arithmetic function of one variable. The asymptotic behaviour of  $\sum_{n \leq x} f(n)$  is a classic problem of analytic number theory, deeply studied for various specific functions and classes. Let us consider the problem of estimating of  $\sum_{m,n \leq x} f(mn)$ .

The divisor function  $\tau$  is a simple, but non-trivial case. Applying Busche—Ramanujan identity

$$(1) \quad \tau(mn) = \sum_{d|\gcd(m,n)} \tau(m/d)\tau(n/d)\mu(d)$$

we split variables and obtain

$$\sum_{m,n \leq x} \tau(mn) = \sum_{\substack{j,k,l \\ j,k \leq x/l}} \tau(j)\tau(k)\mu(l) = \sum_{l \leq x} \mu(l) \left( \sum_{j \leq x/l} \tau(j) \right)^2.$$

Using Huxley's estimate [4]  $\sum_{j \leq y} \tau(j) = y \log y + (2\gamma - 1)y + O(y^{\theta+\varepsilon})$ , where  $\theta = 131/416$ , we regroup terms and get

$$(2) \quad \sum_{m,n \leq x} \tau(mn) = x^2 \left( \left( \sum_{l=1}^{\infty} \frac{\mu(l)}{l^2} \right) \left( \log^2 x + 2(2\gamma - 1) \log x + (2\gamma - 1)^2 \right) - \right. \\ \left. - \left( \sum_{l=1}^{\infty} \frac{\mu(l) \log l}{l^2} \right) \left( 2 \log x + 2(2\gamma - 1) \right) + \sum_{l=1}^{\infty} \frac{\mu(l) \log^2 l}{l^2} \right) + O(x^{1+\theta+\varepsilon}).$$

It is natural to ask whether the main term can be derived analytically, by complex integration method. We will not go into details, but note that

$$\sum_{a,b=0}^{\infty} \tau(p^{a+b}) x^a y^b = \sum_{a,b=0}^{\infty} (a+b+1) x^a y^b = \frac{1-xy}{(1-x)^2(1-y)^2}, \quad |x|, |y| < 1.$$

The series  $\sum_{m,n=1}^{\infty} \tau(mn) m^{-z} n^{-w}$  converges absolutely for  $\Re z, \Re w > 1$ , so by multiplicativity in this region we have

$$(3) \quad \sum_{m,n=1}^{\infty} \frac{\tau(mn)}{m^z n^w} = \prod_p \sum_{a,b=0}^{\infty} \frac{\tau(p^{a+b})}{p^{az+bw}} = \prod_p \frac{1-p^{-z-w}}{(1-p^{-z})^2(1-p^{-w})^2} = \frac{\zeta^2(z)\zeta^2(w)}{\zeta(z+w)}.$$

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Achieved representation allows to compute the coefficient of multiple Laurent series for  $x^{z+w}z^{-1}w^{-1}\sum_{m,n=1}^{\infty}\tau(mn)m^{-z}n^{-w}$  at  $1/(z-1)(w-1)$ , which appears coinciding with the main term of (2).

Our paper is devoted to

$$\sum_{m,n \leq x} \tau_{1,2}(mn),$$

where  $\tau_{1,2}(n) = \sum_{ab^2=n} 1$ . This function is not as lucky as  $\tau$  and does not possess representation like (1), so there is no easy way to split  $m$  and  $n$ .

The main result is

**Theorem 1.**

$$\sum_{m,n \leq x} \tau_{1,2}(mn) = C_1 x^2 + C_2 x^{3/2} + O(x^{10/7+\varepsilon}),$$

where  $C_1 = 2.995\dots$ ,  $C_2 = -5.404\dots$  are computable constants.

This theorem is analogous to the estimate by Graham and Kolesnik [2]

$$\sum_{n \leq x} \tau_{1,2}(n) = \zeta(2)x + \zeta(1/2)x^{1/2} + O(x^{\beta+\varepsilon}), \quad \beta = 1057/4785 \approx 0.2209.$$

## 2. NOTATIONS

Letter  $p$  with or without indexes denotes a prime number. We write  $f \star g$  for the Dirichlet convolution

$$(f \star g)(n) = \sum_{d|n} f(d)g(n/d).$$

In asymptotic relations we use  $\sim$ ,  $\asymp$ , Landau symbols  $O$  and  $o$ , Vinogradov symbols  $\ll$  and  $\gg$  in their usual meanings. All asymptotic relations are given as an argument (usually  $x$ ) tends to the infinity.

Letter  $\gamma$  denotes Euler–Mascheroni constant. Everywhere  $\varepsilon > 0$  is an arbitrarily small number (not always the same even in one equation).

As usual  $\zeta(s)$  is the Riemann zeta-function. Real and imaginary components of the complex  $s$  are denoted as  $\sigma := \Re s$  and  $t := \Im s$ , so  $s = \sigma + it$ .

For a fixed  $\sigma \in [1/2, 1]$  define

$$\mu(\sigma) := \limsup_{t \rightarrow \infty} \frac{\log |\zeta(\sigma + it)|}{\log t}.$$

## 3. PRELIMINARY ESTIMATES

We say that a function is symmetric if any permutation of arguments does not change its value.

Let  $f$  be an arithmetic function of  $r$  variables. The associated Dirichlet series are defined as

$$F(s_1, \dots, s_r) = \sum_{n_1, \dots, n_r=1}^{\infty} f(n_1, \dots, n_r) n_1^{-s_1} \dots n_r^{-s_r}$$

and a tuple  $(\sigma_1, \dots, \sigma_r)$  is called abscissas of absolute convergence if  $F(s_1, \dots, s_r)$  converges absolutely in the region  $\Re s_1 > \sigma_1, \dots, \Re s_r > \sigma_r$ .

**Lemma 1.** *Let  $f$  be a symmetric arithmetic function of  $r$  variables and  $(\sigma_a, \dots, \sigma_a)$  are abscissas of absolute convergence of the associated Dirichlet series  $F(s_1, \dots, s_r)$ . Define*

$$(4) \quad F_r^\heartsuit(\sigma, x, T) := \sum_{n_1, \dots, n_r=1}^{\infty} \frac{|f(n_1, \dots, n_r)|(n_1 \dots n_r)^{-\sigma}}{\min_{j=1, \dots, r} (T |\log(x/n_j)| + 1)}.$$

and let

$$(5) \quad \sum'_{n_1, \dots, n_r \leq x} f(n_1, \dots, n_r) := \sum_{n_1, \dots, n_r \leq x} f(n_1, \dots, n_r) h(x/n_1) \cdots h(x/n_r),$$

where  $h(y) = 0$  for  $0 < y < 1$ ,  $h(1) = 1/2$  and  $h(y) = 1$  otherwise.

For  $x \geq 2$ ,  $T \geq 2$ ,  $\sigma \leq \sigma_a$ ,  $\delta > 0$ ,  $\kappa = \sigma_a - \sigma + \delta/\log x$ ,  $1 = N_1 \leq \dots \leq N_r$ ,  $1 = M_1 \leq \dots \leq M_r$  and  $N_0 := N_1 + \dots + N_r$  we have

$$(6) \quad \left| \sum'_{n_1, \dots, n_r \leq x} \frac{f(n_1, \dots, n_r)}{(n_1 \cdots n_r)^s} - \frac{1}{(2\pi i)^r} \int_{N_1 \kappa - iM_1 T}^{N_1 \kappa + iM_1 T} \cdots \int_{N_r \kappa - iM_r T}^{N_r \kappa + iM_r T} F(s + w_1, \dots, s + w_r) x^{w_1 + \dots + w_r} \frac{dw_1 \cdots dw_r}{w_1 \cdots w_r} \right| \ll x^{N_0(\sigma_a - \sigma)} F_r^\heartsuit(\sigma_a + \delta/\log x, x, T).$$

*Proof.* This is a result of Balazard, Naimi and Pétermann [1, Prop. 6].  $\square$

**Lemma 2.** Let  $f(t) \geq 0$ . If

$$\int_1^T f(t) dt \ll g(T),$$

where  $g(T) = T^\alpha \log^\beta T$ ,  $\alpha \geq 1$ , then

$$I(T) := \int_1^T \frac{f(t)}{t} dt \ll \begin{cases} \log^{\beta+1} T & \text{if } \alpha = 1, \\ T^{\alpha-1} \log^\beta T & \text{if } \alpha > 1. \end{cases}$$

*Proof.* Let us divide the interval of integration into parts:

$$\begin{aligned} I(T) &\leq \sum_{k=0}^{\lfloor \log_2 T \rfloor - 1} \int_{T/2^{k+1}}^{T/2^k} \frac{f(t)}{t} dt + g(2) < \\ &< \sum_{k=0}^{\log_2 T} \frac{1}{T/2^{k+1}} \int_1^{T/2^k} f(t) dt + g(2) \ll \sum_{k=0}^{\lfloor \log_2 T \rfloor - 1} \frac{g(T/2^k)}{T/2^{k+1}}. \end{aligned}$$

Now the lemma's statement follows from elementary estimates.  $\square$

**Lemma 3.** Let  $\eta > 0$  be arbitrarily small. Then for growing  $|t| \geq 3$

$$(7) \quad \zeta(s) \ll \begin{cases} |t|^{1/2 - (1 - 2\mu(1/2))\sigma}, & \sigma \in [0, 1/2], \\ |t|^{2\mu(1/2)(1-\sigma)}, & \sigma \in [1/2, 1 - \eta], \\ |t|^{2\mu(1/2)(1-\sigma)} \log^{2/3} |t|, & \sigma \in [1 - \eta, 1], \\ \log^{2/3} |t|, & \sigma \in [1, 1 + \eta], \\ 1, & \sigma \geq 1 + \eta. \end{cases}$$

*Proof.* Estimates follow from Phragmén–Lindelöf principle and estimates of  $\zeta(s)$  at  $\sigma = 0, 1/2, 1$ . See Titchmarsh [7, Ch. 5] or Ivić [5, Ch. 7.5] for details.  $\square$

**Lemma 4.**

$$\int_1^T |\zeta(\sigma + it)|^2 dt \ll T, \quad 1/2 < \sigma < 1.$$

*Proof.* See Ivić [5, (1.76)].  $\square$

## 4. REDUCTION TO COMPLEX INTEGRATION

Applying Lemma 1 with  $r = 2$ ,  $f(n_1, n_2) = \tau_{1,2}(n_1 n_2)$ ,  $\sigma = s = 0$ ,  $\sigma_a = 1$ ,  $N_1 = N_2 = M_1 = M_2 = 1$ ,  $\delta = 1$ ,  $\log T \asymp \log x$  and writing  $(m, n, z, w, c)$  instead of  $(n_1, n_2, w_1, w_2, \kappa)$  for convenience we deduce from (6) that

$$(8) \quad \sum'_{m, n \leq x} \tau_{1,2}(mn) = \frac{1}{(2\pi i)^2} \iint_{[c-iT, c+iT]^2} F(z, w) \frac{x^{z+w}}{zw} dz dw + O\left(x^2 F_2^\heartsuit(c, x, T)\right),$$

where  $c = 1 + 1/\log x$  and

$$(9) \quad F(z, w) = \sum_{m, n=1}^{\infty} \frac{\tau_{1,2}(mn)}{m^z n^w}, \quad \Re z, \Re w > 1.$$

By (4) for non-integer  $x$

$$(10) \quad TF_2^\heartsuit(c, x, T) \ll \sum_{m, n} \frac{\tau_{1,2}(mn)}{(mn)^c \min(|\log \frac{x}{n}|, |\log \frac{x}{m}|)} \ll \sum_{\substack{|\log \frac{x}{n}| \geq 1 \\ |\log \frac{x}{m}| \geq 1}} \frac{\tau_{1,2}(mn)}{(mn)^c} + \\ + \sum_{\substack{|\log \frac{x}{n}| \leq 1 \\ |\log \frac{x}{m}| \geq 1}} \frac{\tau_{1,2}(mn)}{(mn)^c |\log \frac{x}{n}|} + \sum_{\substack{|\log \frac{x}{n}| \leq 1 \\ |\log \frac{x}{m}| \leq 1}} \frac{\tau_{1,2}(mn)}{(mn)^c \min(|\log \frac{x}{n}|, |\log \frac{x}{m}|)} := \\ := \Sigma_1 + \Sigma_2 + \Sigma_3.$$

We have  $\Sigma_1 \ll \sum_{m, n=1}^{\infty} \tau_{1,2}(mn)/(mn)^c = F(c, c)$  and we will show below in (19) that

$$(11) \quad F(c, c) \ll \frac{1}{(c-1)^2} = \log^2 x.$$

Further, for  $x$  such that  $|\log \frac{x}{n}| \leq 1$  we have  $|\log \frac{x}{n}| \geq c|x-n|/x$  for  $c = 1/(e-1)$ . Then

$$\Sigma_2 \ll \sum_{x/e \leq n \leq xe} \sum_m \frac{\tau_{1,2}(mn)x}{(mn)^c |x-n|}.$$

Note that  $\tau_{1,2}(mn) \leq \tau(mn) \leq \tau(m)\tau(n)$ , because  $\tau$  is completely submultiplicative. Thus

$$\Sigma_2 \ll x \sum_{x/e \leq n \leq xe} \frac{\tau(n)}{n^c |x-n|} \sum_m \frac{\tau(m)}{m^c}.$$

Here

$$\sum_{m=1}^{\infty} \tau(m) m^{-c} = \zeta^2(c) \ll (c-1)^{-2} = \log^2 x.$$

Let  $M(y) = \max_{n \leq y} \tau(n)$ . We have

$$\Sigma_2 \ll x M(xe) \log^2 x \sum_{x/e \leq n \leq xe} \frac{1}{n^c |x-n|},$$

where the last sum is  $\ll x^{-c} \log x \ll x^{-1} \log x$ , so finally

$$(12) \quad \Sigma_2 \ll M(xe) \log^3 x.$$

Now consider  $\Sigma_3$ . Defining  $M_{1,2}(y) = \max_{n \leq y} \tau_{1,2}(n)$  we obtain

$$(13) \quad \Sigma_3 \ll \sum_{x/e \leq n \leq m \leq xe} \frac{\tau_{1,2}(mn)x}{(mn)^c \min(|x-n|, |x-m|)} \ll \\ \ll \frac{xM_{1,2}(x^2e^2)}{x^{2c}} \sum_{x/e \leq n \leq m \leq xe} \max(|x-n|^{-1}, |x-m|^{-1}) \ll M_{1,2}(x^2e^2) \log x.$$

Standard estimates [3, Th. 315] give  $M_{1,2}(y) \leq M(y) \ll y^\varepsilon$ , so substituting (11), (12) and (13) into (10) we obtain

$$(14) \quad F_2^\heartsuit(c, x, T) \ll T^{-1} (M(xe) \log^3 x + M_{1,2}(x^2e^2) \log x) \ll T^{-1} x^\varepsilon.$$

Note also that by definition (5)

$$(15) \quad \left| \sum_{m, n \leq x} \tau_{1,2}(mn) - \sum'_{m, n \leq x} \tau_{1,2}(mn) \right| \ll \sum_{n \leq x} \tau_{1,2}(\lfloor x \rfloor n) \ll M(x^2)x.$$

Combining (8), (14) and (15) we get

$$(16) \quad \sum_{m, n \leq x} \tau_{1,2}(mn) = \frac{1}{(2\pi i)^2} \iint_{[c-iT, c+iT]^2} F(z, w) \frac{x^{z+w}}{zw} dz dw + \\ + O(x^{1+\varepsilon} + T^{-1}x^{2+\varepsilon}).$$

## 5. DOUBLE DIRICHLET SERIES FOR $\tau_{1,2}$

Let us return to (9) and extract a product of zeta-functions from  $F(z, w)$ . Define

$$(17) \quad f(x, y) = \sum_{a, b=0}^{\infty} \tau_{1,2}(p^{a+b}) x^a y^b, \quad |x|, |y| < 1.$$

Using identity

$$\tau_{1,2}(p^a) - \tau_{1,2}(p^{a-1}) - \tau_{1,2}(p^{a-2}) + \tau_{1,2}(p^{a-3}) = 0$$

multiply both sides of (17) by  $(1-x)(1-x^2)$ :

$$(1-x)(1-x^2)f(x, y) = \\ = \sum_{a=3}^{\infty} \sum_{b=0}^{\infty} (\tau_{1,2}(p^{a+b}) - \tau_{1,2}(p^{a+b-1}) - \tau_{1,2}(p^{a+b-2}) + \tau_{1,2}(p^{a+b-3})) x^a y^b + \\ + \sum_{b=0}^{\infty} y^b ((1-x-x^2)\tau_{1,2}(p^b) + (1-x)\tau_{1,2}(p^{b+1})x + \tau_{1,2}(p^{b+2})x^2) = \\ = \sum_{b=0}^{\infty} y^b ((1-x-x^2)\tau_{1,2}(p^b) + (x-x^2)\tau_{1,2}(p^{b+1})x + x^2\tau_{1,2}(p^{b+2}))$$

and further

$$(1-x)(1-x^2)(1-y)(1-y^2)f(x, y) = \\ = (1-x-x^2)((1-y-y^2) + (1-y)y + 2y^2) + \\ + (x-x^2)((1-y-y^2) + 2(1-y)y + 2y^2) + \\ + x^2(2(1-y-y^2) + 2(1-y)y + 3y^2) = \\ = 1 + xy - x^2y - xy^2,$$

which induces

$$(18) \quad f(x, y) = \frac{1 + xy - x^2y - xy^2}{(1-x)(1-x^2)(1-y)(1-y^2)} = \frac{1 - x^2y - xy^2 - x^2y^2 + x^3y^2 + x^2y^3}{(1-x)(1-x^2)(1-y)(1-y^2)(1-xy)}.$$

Representation (18) immediately implies that

$$(19) \quad F(z, w) = \prod_p f(p^{-z}, p^{-w}) = \zeta(z)\zeta(2z)\zeta(w)\zeta(2w)\zeta(z+w)G(z, w) = \frac{\zeta(z)\zeta(2z)\zeta(w)\zeta(2w)\zeta(z+w)}{\zeta(2z+w)\zeta(2w+z)}H(z, w),$$

where series  $H(z, w)$  converges absolutely in the region  $\Re(2z + 2w) > 1$ . Definitely  $G(z, w)$  converges absolutely for  $(z, w) \in Q := \{\Re z \geq 1/3, \Re w \geq 1/3\}$ .

Product of zeta-functions (19) shows that inside of the region  $Q$  function  $F(z, w)$  has poles along lines  $z = 1$ ,  $z = 1/2$ ,  $w = 1$ ,  $w = 1/2$  and  $z + w = 1$ . All of them are of the first order, except poles at  $(1, 1)$ ,  $(1, 1/2)$ ,  $(1/2, 1)$ , which are of the second order, and a pole at  $(1/2, 1/2)$ , which is of the third order.

Both (3) and (19) are partial cases of a general rule, which will be stated as a lemma.

**Lemma 5.** *Let  $\tau_{1,k}(n) = \sum_{ab^k=n} 1$ . Then for  $\Re z, \Re w > 1$  we have*

$$(20) \quad \sum_{m,n=1}^{\infty} \frac{\tau_{1,k}(mn)}{m^z n^w} = \zeta(z)\zeta(w) \frac{\prod_{l=0}^k \zeta(lz + (k-l)w)}{\prod_{l=1}^k \zeta(lz + (k+1-l)w)} H_k(z, w),$$

where the series  $H_k$  converges absolutely for  $\Re z, \Re w > 1/(k+2)$ .

*Proof.* Cases  $k = 1$  and  $k = 2$  has been proven above, so we consider  $k > 2$  only. Let

$$f(x, y) = \sum_{a,b=0}^{\infty} \tau_{1,k}(p^{a+b}) x^a y^b, \quad |x|, |y| < 1.$$

For a monomial  $M$  let  $[M]f(x, y)$  be a coefficient at  $M$  in the series  $f$ . Here

$$[x]f(x, y) = [y]f(x, y) = \tau_{1,k}(p) = 1,$$

so let us define

$$\begin{aligned} g(x, y) &= (1-x)(1-y)f(x, y) = \\ &= \sum_{a,b=1}^{\infty} (\tau_{1,k}(p^{a+b}) - 2\tau_{1,k}(p^{a+b-1}) + \tau_{1,k}(p^{a+b-2})) x^a y^b + \\ &\quad + \sum_{a=1}^{\infty} (\tau_{1,k}(p^a) - \tau_{1,k}(p^{a-1})) (x^a + y^a) + 1. \end{aligned}$$

We have

$$\tau_{1,k}(p^a) = \begin{cases} 1, & a < k, \\ 2, & k \leq a < 2k, \end{cases}$$

so one can verify that

$$[x^a y^b]g(x, y) = \begin{cases} 0, & a+b < k, \\ 1, & a+b = k, \\ 0, & a+b = k+1, \quad ab = 0 \\ -1, & a+b = k+1, \quad ab > 0. \end{cases}$$

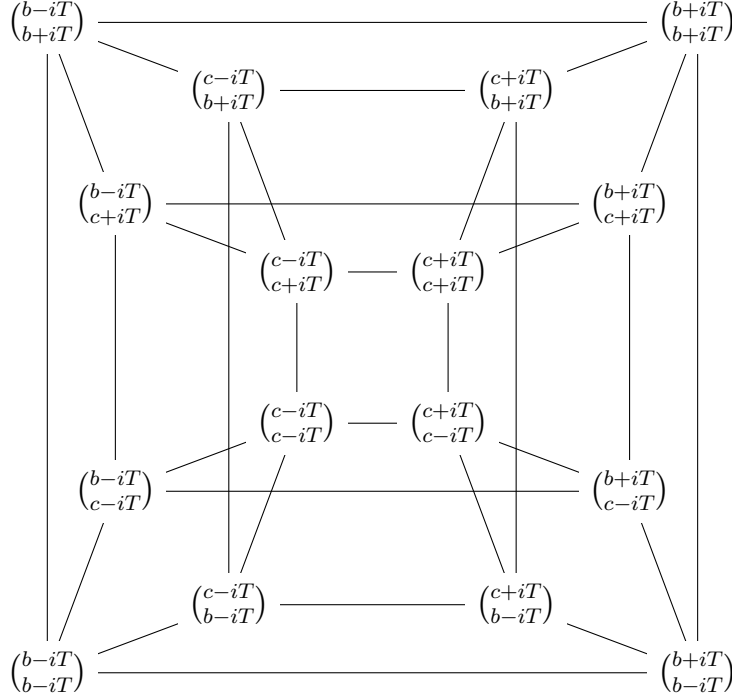


FIGURE 1. The hyperrectangle  $R$  with opposite vertices  $(b-iT, b-iT)$  and  $(c+iT, c+iT)$

Thus

$$f(x, y) = \frac{1}{(1-x)(1-y)} \frac{\prod_{l=1}^k (1 - x^l y^{k+1-l})}{\prod_{l=0}^k (1 - x^l y^{k-l})} h(x, y),$$

where all monomials of the series  $h(x, y)$  has degree at least  $k+2$ .  $\square$

## 6. PATH OF INTEGRATION AND THE MAIN TERM

Our aim is to translate the domain of integration in (16) from  $[c-iT, c+iT]^2$  till  $[b-iT, b+iT]^2$ , where  $b = 1/3$ . This is trickier than translating in the one-dimensional case, because a hyperrectangle  $R$  with opposite vertices  $(b-iT, b-iT)$  and  $(c+iT, c+iT)$  has 24 two-dimensional faces. Figure 1 contains a schematic plain projection of  $R$  with 16 vertices and 32 edges marked.

Denote  $L(z, w) = G(z, w)x^{z+w}z^{-1}w^{-1}$ . This function has the same poles in  $R$  as  $G(z, w)$  has. Note that (on contrary with integration by one-dimensional contour) poles of the first order do not induce divergence of integrals by plane domains: e. g.,  $\iint_{x^2+y^2 \leq 1} \frac{dx dy}{\sqrt{x^2+y^2}} = 2\pi < \infty$ , however  $\int_{x^2 \leq 1} \frac{dx}{x} = \infty$ . Only poles of the second and higher orders are worth to pay attention.

Let  $E(x)$  be the integral of  $L(z, w)$  over all faces of  $R$  except  $[c-iT, c+iT]^2$ . By residue theorem [6]

$$(21) \quad \frac{1}{(2\pi i)^2} \iint_{[c-iT, c+iT]^2} L(z, w) dz dw = \\ = \left( \operatorname{res}_{z=w=1} + \operatorname{res}_{\substack{z=1 \\ w=1/2}} + \operatorname{res}_{\substack{z=1/2 \\ w=1}} + \operatorname{res}_{z=w=1/2} \right) L(z, w) + O(E(x)).$$

Expanding  $L(z, w)$  into Laurent series in two variables we get

$$(22) \quad \operatorname{res}_{z=w=1} L(z, w) = \zeta^3(2)G(1, 1)x^2,$$

$$(23) \quad \operatorname{res}_{\substack{z=1 \\ w=1/2}} L(z, w) = \operatorname{res}_{\substack{z=1/2 \\ w=1}} L(z, w) = \zeta(2)\zeta(\tfrac{1}{2})\zeta(\tfrac{3}{2})G(1, \tfrac{1}{2})x^{3/2},$$

$$(24) \quad \operatorname{res}_{z=w=1/2} L(z, w) \ll x \log x.$$

After substitution into (16) the residue at  $(1/2, 1/2)$  will be absorbed by error term, so it is enough to have only upper bound. Inserting (22), (23) and (24) into (21) we get

$$(25) \quad \frac{1}{(2\pi i)^2} \iint_{[c-iT, c+iT]^2} L(z, w) dz dw = C_1 x^2 + C_2 x^{3/2} + O(x \log x + E(x)),$$

where

$$C_1 = \frac{\pi^6}{216}G(1, 1), \quad C_2 = \frac{\pi^2}{3}\zeta(\tfrac{1}{2})\zeta(\tfrac{3}{2})G(1, \tfrac{1}{2}).$$

Let us calculate numerical values of  $C_1$  and  $C_2$ . Applying formal identity

$$\frac{F(z, w)}{\zeta(z)\zeta(w)} = \prod_p (1 - p^{-z})(1 - p^{-w}) \sum_{a,b=0}^{\infty} \frac{\tau_{1,2}(p^{a+b})}{p^{a+b}}$$

at  $z = w = 1$  we get

$$C_1 = \operatorname{res}_{z=w=1} L(z, w) = \prod_p (1 - p^{-1})^2 \sum_{a,b=0}^{\infty} \frac{\tau_{1,2}(p^{a+b})}{p^{a+b}} = 2.995 \dots$$

The product converges absolutely because

$$(1 - p^{-1})^2 \sum_{a,b=0}^{\infty} \frac{\tau_{1,2}(p^{a+b})}{p^{a+b}} = (1 - 2p^{-1} + O(p^{-2}))(1 + 2p^{-1} + O(p^{-2})) = 1 + O(p^{-2}).$$

Similarly

$$\frac{F(z, w)}{\zeta(z)\zeta(w)\zeta(2z)} = \prod_p (1 - p^{-z})(1 - p^{-w})(1 - p^{-2z}) \sum_{a,b=0}^{\infty} \frac{\tau_{1,2}(p^{a+b})}{p^{a+b}}$$

implies

$$C_2 = 2 \operatorname{res}_{\substack{z=1 \\ w=1/2}} L(z, w) = 2\zeta(1/2) \prod_p (1 - p^{-1})^2 (1 - p^{-1/2}) \sum_{a,b=0}^{\infty} \frac{\tau_{1,2}(p^{a+b})}{p^{a+b/2}} = -5.404 \dots$$

## 7. THE ERROR TERM

Let us estimate  $E(x)$ . It was defined above to consist of integrals over 23 of 24 faces of the hyperrectangle  $R$ , but due to the symmetry many of these integrals can be estimated in the same way.

In computations below we assume  $x^{1/2} \ll T \ll x$ , the exact value of  $T$  will be specified later in (28).

There are 2 faces of form  $[b - iT, b + iT] \times [c - iT, c + iT]$ . We have

$$\begin{aligned} I_1 &:= \int_{b-iT}^{b+iT} \int_{c-iT}^{c+iT} L(z, w) dz dw \ll \iint_{[1, T]^2} \zeta(b + it_1) \zeta(2b + 2it_1) \times \\ &\quad \times \zeta(c + it_2) \zeta(2c + 2it_2) \zeta(b + c + i(t_1 + t_2)) x^{b+c} t_1^{-1} t_2^{-1} dt_1 dt_2. \end{aligned}$$

By (7) we can estimate

$$\zeta(c + it_2) \zeta(2c + 2it_2) \zeta(b + c + i(t_1 + t_2)) \ll \log^{2/3} T \cdot 1 \cdot 1.$$



As soon as  $x^{1/\log x} \ll 1$  we have  $x^{b+c} \ll x^{4/3}$ . Also  $\int_1^T t_2^{-1} dt_2 \ll \log T$ . Thus  $I_1$  can be estimated as

$$I_1 \ll x^{4/3} \log^{5/3} T \int_1^T \zeta(b+it) \zeta(2b+2it) t^{-1} dt.$$

By functional equation for  $\zeta$ , Lemma 4 and Lemma 2

$$(26) \quad J := \int_1^T \zeta(b+it) \zeta(2b+2it) t^{-1} dt \ll \int_1^T t^{1/6} \zeta^2(2/3+it) t^{-1} dt \ll T^{1/6} \log T.$$

Then

$$(27) \quad I_1 \ll x^{4/3} T^{1/6} \log^{8/3} T.$$

We will show below in (40) that integrals over other faces (and so  $E(x)$  as a whole) are less than either  $I_1$  or  $x^{2+\varepsilon} T^{-1}$ , so  $T$  should be chosen to equalize this two magnitudes:

$$(28) \quad T = x^{4/7}.$$

Substitute it into (16) and (25) to obtain the final error term  $x^{10/7+\varepsilon}$ , which improves the statement of the Theorem 1.

From here and till the end of the section we will omit factors  $\ll x^\varepsilon$  in asymptotic estimates for the brevity: they do not influence the resulting error term.

There are 4 faces of form  $[b-iT, b+iT] \times [b \pm iT, c \pm iT]$ . We have

$$\begin{aligned} I_2 &:= \int_{b-iT}^{b+iT} \int_{b+iT}^{c+iT} L(z, w) dz dw \ll \int_1^T \int_b^c \zeta(b+it) \zeta(2b+2it) \times \\ &\quad \times \zeta(\sigma+iT) \zeta(2\sigma+2iT) \zeta(b+\sigma+i(t+T)) x^{b+\sigma} t^{-1} T^{-1} d\sigma dt \ll \\ &\ll x^{1/3} J T^{-1} \max_{\substack{\sigma \in [b, c] \\ t \in [1, T]}} \zeta(\sigma+iT) \zeta(2\sigma+2iT) \zeta(b+\sigma+i(t+T)) x^\sigma \ll \\ &\ll x^{1/3} T^{-5/6} \max_{\sigma \in [b, c]} \zeta(\sigma+iT) \zeta(\sigma+1/3+iT) \zeta(2\sigma+iT) x^\sigma. \end{aligned}$$

Splitting  $[b, c]$  into intervals  $[1/3, 1/2]$ ,  $[1/2, 2/3]$ ,  $[2/3, c]$  and estimating  $\zeta(\sigma+iT) \times \zeta(\sigma+1/3+iT) \zeta(2\sigma+iT) x^\sigma$  on each of them separately, we get

$$I_2 \ll x^{1/3} T^{-5/6} (T^{\mu(1/3)+2\mu(2/3)} x^{1/2} + T^{\mu(1/2)+\mu(5/6)} x^{2/3} + T^{\mu(2/3)} x).$$

Utilizing rough estimate  $\mu(1/2) \leq 1/6$  from [7, Th. 5.5] we get by (7) that

$$(29) \quad \mu(\sigma) \leq \begin{cases} 1/2 - 2\sigma/3, & \sigma \in [0, 1/2], \\ (1-\sigma)/3, & \sigma \in [1/2, 1] \end{cases}$$

and

$$(30) \quad \mu(1/3) \leq 5/18, \quad \mu(2/3) \leq 1/9, \quad \mu(5/6) \leq 1/18,$$

so

$$(31) \quad I_2 \ll x^{1/3} T^{-5/6} (T^{1/2} x^{1/2} + T^{2/9} x^{2/3} + T^{1/9} x) \ll x^{4/3}.$$

There is 1 face of form  $[b - iT, b + iT]^2$ . Applying (30) we have

$$\begin{aligned} I_3 &:= \iint_{[b-iT, b+iT]^2} L(z, w) dz dw \ll \iint_{[1, T]^2} \zeta(b + it_1) \zeta(2b + 2it_1) \times \\ &\quad \times \zeta(b + it_2) \zeta(2b + 2it_2) \zeta(2b + i(t_1 + t_2)) x^{2b} t_1^{-1} t_2^{-1} dt_1 dt_2 \ll \\ &\quad \ll x^{2/3} \iint_{[1, T]^2} t_1^{5/18+1/9-1} t_2^{5/18+1/9-1} (t_1 + t_2)^{1/9} dt_1 dt_2, \end{aligned}$$

which implies

$$(32) \quad I_3 \ll x^{2/3} T^{8/9},$$

which is less than  $x^{4/3}$  by our choice of  $T$  in (28).

There are 4 faces of form  $[c - iT, c + iT] \times [b \pm iT, c \pm iT]$ . We have

$$\begin{aligned} (33) \quad I_4 &:= \int_{c-iT}^{c+iT} \int_{b+iT}^{c+iT} L(z, w) dz dw \ll \\ &\ll \int_1^T \int_b^c \zeta(c + it) \zeta(2c + 2it) \zeta(\sigma + iT) \zeta(2\sigma + 2iT) \zeta(c + \sigma + i(t + T)) \times \\ &\quad \times x^{c+\sigma} t^{-1} T^{-1} d\sigma dt \ll x T^{-1} \int_b^c \zeta(\sigma + iT) \zeta(2\sigma + 2iT) x^\sigma d\sigma. \end{aligned}$$

Here

$$\int_b^c \zeta(\sigma + iT) \zeta(2\sigma + 2iT) x^\sigma d\sigma \ll \max_{\sigma \in [b, c]} \zeta(\sigma + iT) \zeta(2\sigma + 2iT) x^\sigma.$$

For  $\sigma \in [b, 1/2]$  we have

$$(34) \quad \zeta(\sigma + iT) \zeta(2\sigma + 2iT) x^\sigma \ll T^{\mu(1/3)+\mu(2/3)} x^{1/2} \ll T x^{1/3}.$$

Taking into account (29) for  $\sigma \in [1/2, 1]$  we get

$$(35) \quad \zeta(\sigma + iT) \zeta(2\sigma + 2iT) x^\sigma \ll T^{\mu(\sigma)} x^\sigma \ll x^{\mu(\sigma)+\sigma} \ll x^{(1+2\sigma)/3} \ll x.$$

Returning to (33) we get

$$(36) \quad I_4 \ll x^2 T^{-1} + x^{4/3}.$$

There are 4 faces of form  $[b \pm iT, c \pm iT]^2$ . We have

$$\begin{aligned} (37) \quad I_5 &:= \iint_{[b+iT, c+iT]^2} L(z, w) dz dw \ll \max_{(z, w) \in [b+iT, c+iT]^2} L(z, w) \ll \\ &\ll \max_{\sigma_1, \sigma_2 \in [b, c]} \zeta(\sigma_1 + iT) \zeta(2\sigma_1 + 2iT) \zeta(\sigma_2 + iT) \zeta(2\sigma_2 + 2iT) \zeta(\sigma_1 + \sigma_2 + 2iT) \times \\ &\quad \times x^{\sigma_1 + \sigma_2} T^{-2} \ll T^{2\mu(1/3)+3\mu(2/3)-2} x^2 \ll x^2 T^{-1}. \end{aligned}$$

Finally, there are 8 faces, which are parallel either to  $z$ - or  $w$ -plane, of form  $[b - iT, c + iT] \times w$ , where  $w \in W := \{b \pm iT, c \pm iT\}$ . We have

$$\begin{aligned} I_6 &:= \iint_{b-iT}^{c+iT} L(z, b + iT) dz \ll \int_1^T \int_b^c \zeta(\sigma + it) \zeta(2\sigma + 2it) \zeta(\sigma + b + i(t + T)) \times \\ &\quad \times \zeta(b + iT) \zeta(2b + 2iT) x^{\sigma+b} t^{-1} T^{-1} d\sigma dt \ll T^{\mu(1/3)+\mu(2/3)-1} x^{1/3} \times \\ &\quad \times \int_1^T \int_b^c \zeta(\sigma + it) \zeta(2\sigma + 2it) \zeta(\sigma + 1/3 + iT) x^\sigma t^{-1} d\sigma dt. \end{aligned}$$

Here

$$\zeta(\sigma + it)\zeta(2\sigma + 2it)\zeta(\sigma + 1/3 + iT)x^\sigma t^{-1} \ll T^{\mu(1/3)+2\mu(2/3)-1}x,$$

so

$$(38) \quad I_6 \ll T^{\mu(1/3)+\mu(2/3)-1}x^{1/3} \int_1^T T^{\mu(1/3)+2\mu(2/3)-1}x \, dt \ll x^{4/3}.$$

Also

$$\begin{aligned} I_7 &:= \iint_{b-iT}^{c+iT} L(z, c+iT) \, dz \ll \int_1^T \int_b^c \zeta(\sigma + it)\zeta(2\sigma + 2it) \times \\ &\quad \times \zeta(\sigma + c + i(t+T))\zeta(c+iT)\zeta(2c+2iT)x^{\sigma+c}t^{-1}T^{-1}d\sigma \, dt \ll \\ &\quad \ll xT^{-1} \int_1^T \int_b^c \zeta(\sigma + it)\zeta(2\sigma + 2it)x^\sigma t^{-1}d\sigma \, dt \end{aligned}$$

We derive from (34) and (35) that

$$\int_b^c \zeta(\sigma + it)\zeta(2\sigma + 2it)x^\sigma d\sigma \ll tx^{1/3} + x,$$

so

$$(39) \quad I_7 \ll xT^{-1} \int_1^T (x^{1/3} + xt^{-1})dt \ll x^2T^{-1} + x^{4/3}.$$

Now summing up (27), (31), (32), (36), (37), (38), (39) we get

$$(40) \quad E(x) \ll x^{4/3}T^{1/6} + x^{2+\varepsilon}T^{-1}.$$

## 8. CONCLUSION

Our result can be slightly improved under the Riemann hypothesis. In such case we have  $\zeta^{\pm 1}(s) \ll x^\varepsilon$  for  $\sigma > 1/2$  and  $\mu(1/2) = 0$  due to [7, (14.2.5)–(14.2.6)]. Then (19) immediately induces  $F(z, w) \ll x^\varepsilon \zeta(z)\zeta(w)$  for  $\Re z, \Re w > 1/4$  and all double integrals, incorporated in  $E(x)$ , can be split and estimated by a product of two one-dimensional integrals. For  $b = 1/4 + 1/\log x$  we obtain

$$\begin{aligned} \int_{b-iT}^{b+iT} \zeta(z) \frac{x^z}{z} dz &\ll x^{1/4+\varepsilon}T^{1/4}, \\ \int_{c-iT}^{c+iT} \zeta(z) \frac{x^z}{z} dz &\ll x^{1+\varepsilon}, \\ \int_{b\pm iT}^{c\pm iT} \zeta(z) \frac{x^z}{z} dz &\ll (x^{1/2+\varepsilon}T^{1/4} + x^{1+\varepsilon})/T. \end{aligned}$$

Then  $E(x) \ll x^{5/4+\varepsilon}T^{1/4}$  and choice  $T = x^{3/5}$  provides us with  $\alpha = 7/5 = 1.4$  in the statement of Theorem 1.

One should expect in the view of (20) that

$$(41) \quad \sum_{m, n \leq x} \tau_{1,k}(mn) = D_1x^2 + D_2x^{1+1/k} + O(x^{\alpha_k+\varepsilon}).$$

Translating the domain of integration till  $[b-iT, b+iT]^2$ , where  $b = 1/(k+1)$ , leads to the error term at least  $x^{\frac{k+2}{k+1}+\varepsilon}T^{\frac{1}{2}-\frac{1}{k+1}} + x^{2+\varepsilon}T^{-1}$ , which corresponds to  $\alpha_k = (4k+2)/(3k+1)$  for the best possible choice of  $T$ . Under the Riemann hypothesis for  $b = 1/2k + 1/\log x$  we obtain  $\alpha_k = (4k-1)/(3k-1)$ . However, for  $k > 2$  both of these estimates are bigger than  $x^{4/3}$  and absorbs the term  $D_2x^{1+1/k}$  in (41). Such result can hardly be reckoned satisfactory.

One can consider the exponential divisor function  $\tau^{(e)}$ , which is multiplicative and defined by  $\tau^{(e)}(p^a) = \tau(a)$ . As far as  $\tau^{(e)}(p^k) = \tau_{1,2}(p^k)$  for  $k = 1, 2, 3, 4$ , the Dirichlet series for  $\tau^{(e)}$  also possesses the representation (19), so Theorem 1 remains valid for  $\tau^{(e)}$  instead of  $\tau_{1,2}$ .

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